Precision calculations in QCD

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Large Hadron Collider Physics Conference
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Precision QCD

Purpose and intention of this talk

Current and future vast LHC precision phenomenology program:

Main challenges to and requirements from precise theory predictions {

- diverse signatures (jets, heavy-quarks, vector bosons, associated productions...)
- larger multiplicites
- higher-order perturbative calculations

How do we go about it and how to improve? Where do we stand and what can we expect next?

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Main focus of this talk: fixed-order predictions in QCD

- methods and frameworks (status and progress)
- (some) pheno results at the current frontier

fixed-order does not give you events but:

- Theoretically well defined framework, improvable (parametrically)
- ingredients for resummation/matching
- analysis of main higher-order effects

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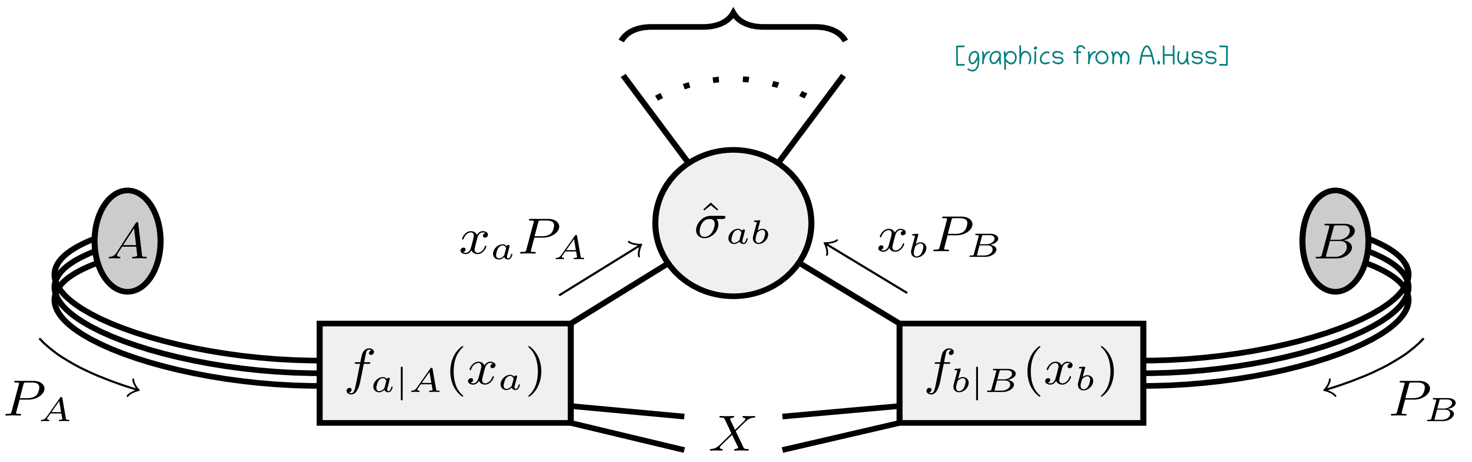
Beyond fixed order QCD and complementary to this talk:

- Broader discussion on current status of pheno studies and events generation: Zanderighi (Tue)
- Jets substructure + flavour: Stagnitto (after this talk)
- Improvements on showers: Herren (Thu)
- generators: Diboson Zanoli, polarised V Pellicciioli (Thu)
- PDFs: Cruz-Martinez (Thu)
- Precise predictions for heavy-quarks: Grazzini (Thu)
- EW + QCDxEW: Lindert (Fri)

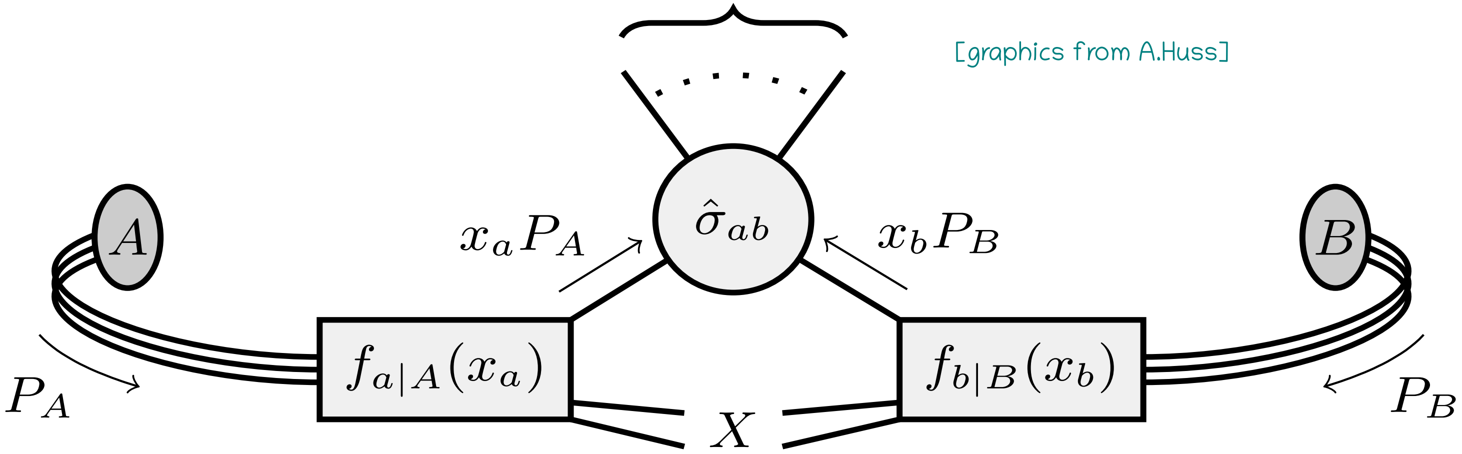
Higher-order QCD predictions



[graphics from A.Huss]

$$\frac{\mathrm{d}\sigma_{AB}}{\mathrm{d}O} = \sum_{ab} \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \, f_{a/A}(x_1) f_{b/B}(x_2) \, \frac{\mathrm{d}\hat{\sigma}_{ab}(x_1, x_2)}{\mathrm{d}O} \times \left(1 + \mathcal{O}\left(\frac{\Lambda^p_{\mathrm{QCD}}}{Q^p}\right)\right)$$

Higher-order QCD predictions



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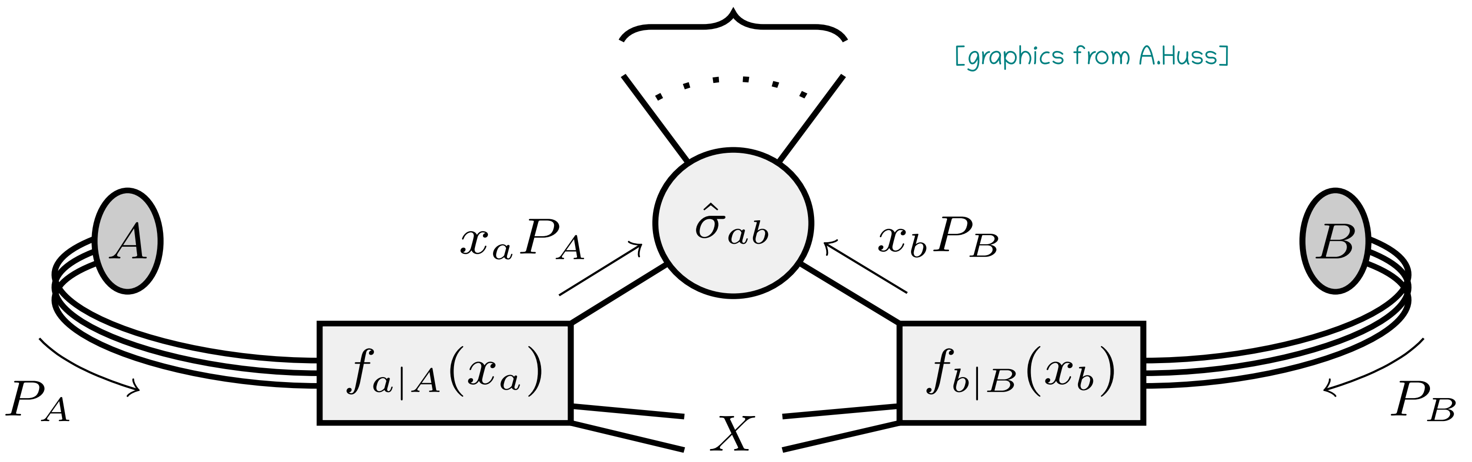
PDFs: non-perturbative

evolution through DGLAP

4-loop splitting functions for N^3LO evolution

Higher-order QCD predictions



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4-loop splitting functions for N^3LO evolution

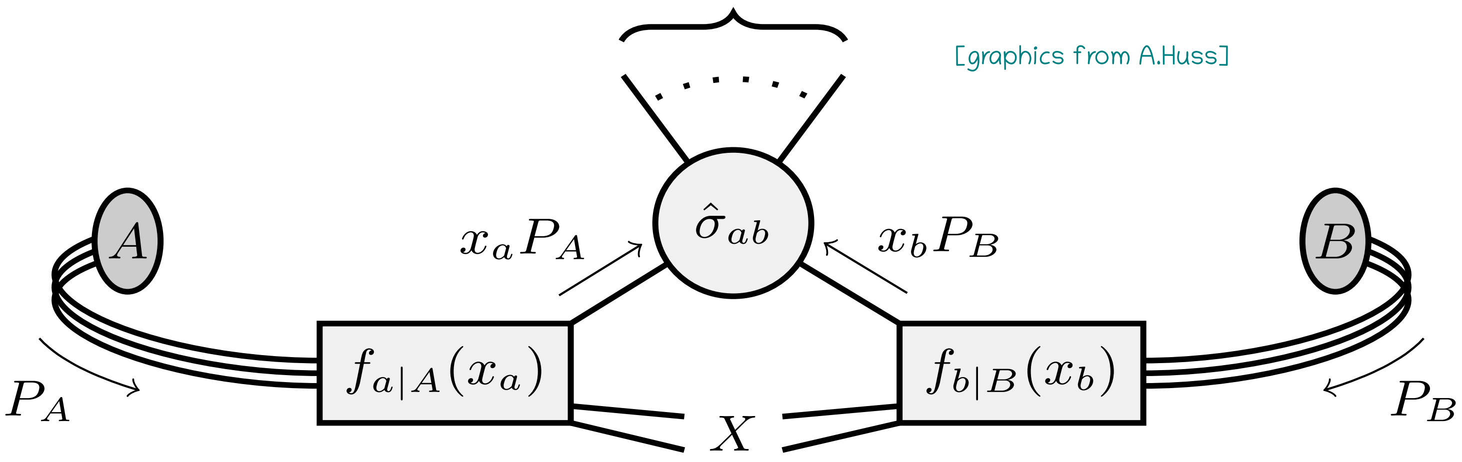
Fully-differential partonic cross-section

Improved parametrically in perturbation theory

Main ingredients to push pert. predictions:

Amplitudes + Subtraction of IR singularities

Higher-order QCD predictions



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Non-perturbative power corrections

Interesting recent progress on pow. corrections for collider observables

see G. Zanderighi (Tue)

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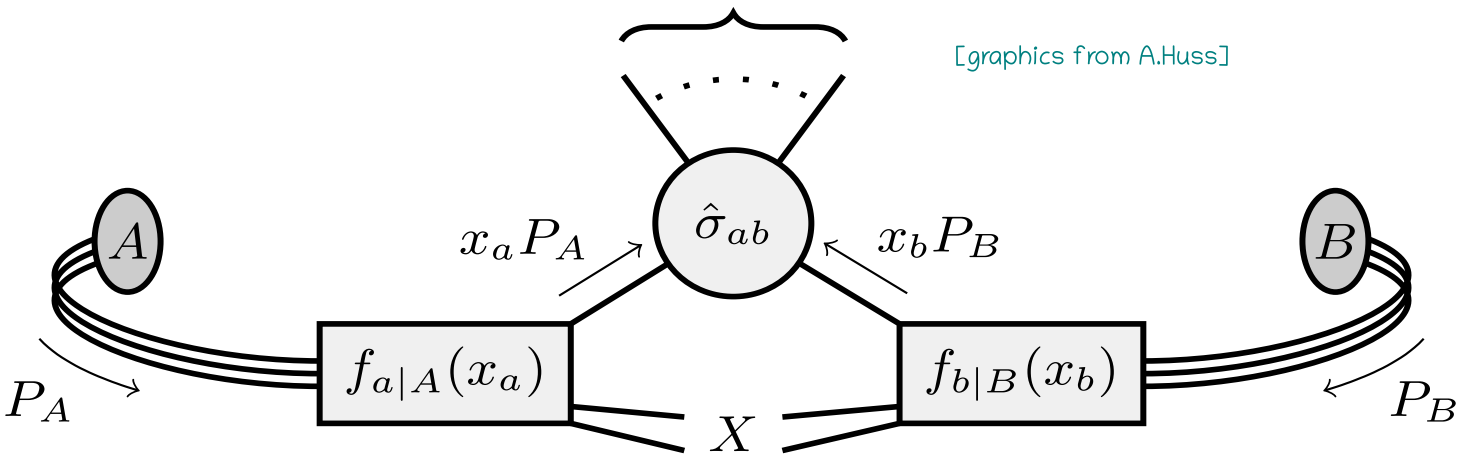

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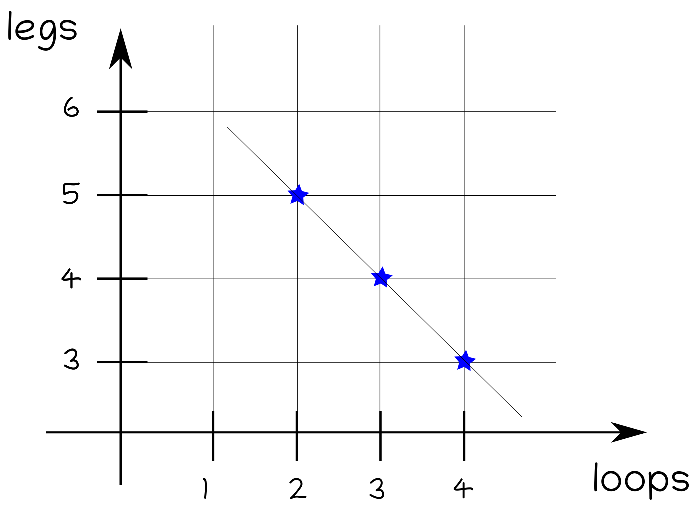
Focus of this talk

Scattering Amplitudes

Status of multiloop scattering ampitudes

Complexity increases fast with loops and scales:
availability of multiscale-multiloop amplitudes are now arguably the bottleneck of NNLO predictions

Current frontier (loops > 1): loops + legs = 7



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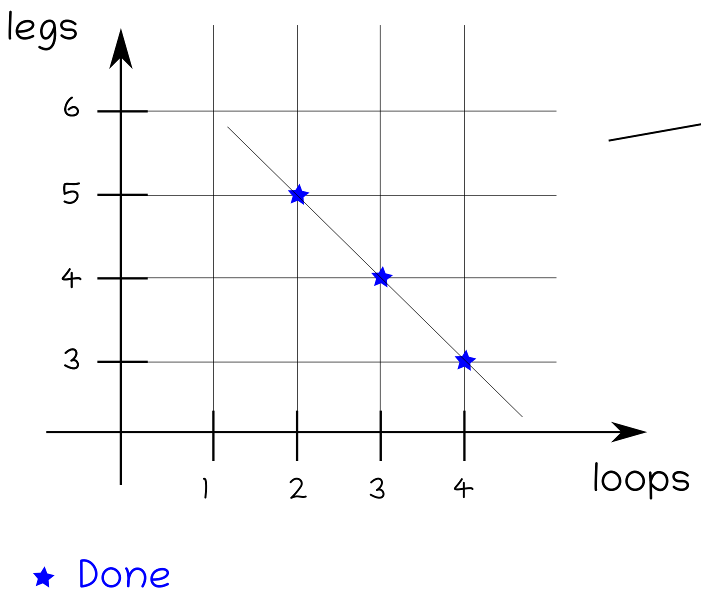
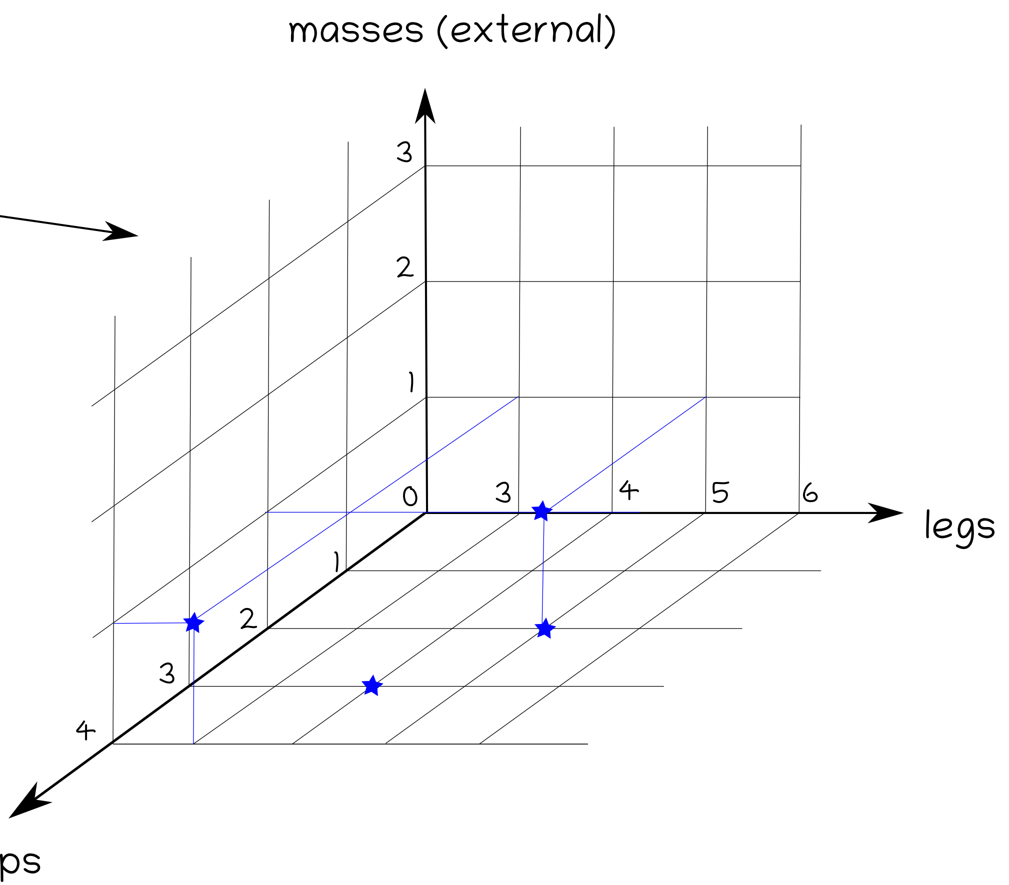


Third direction
in complexity space



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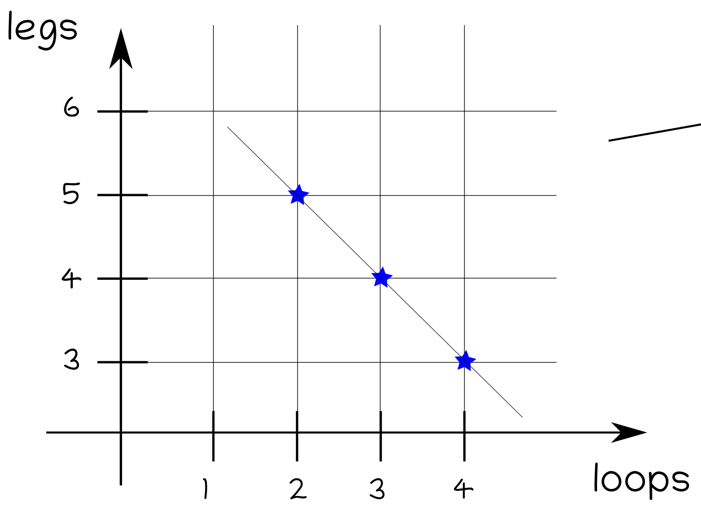

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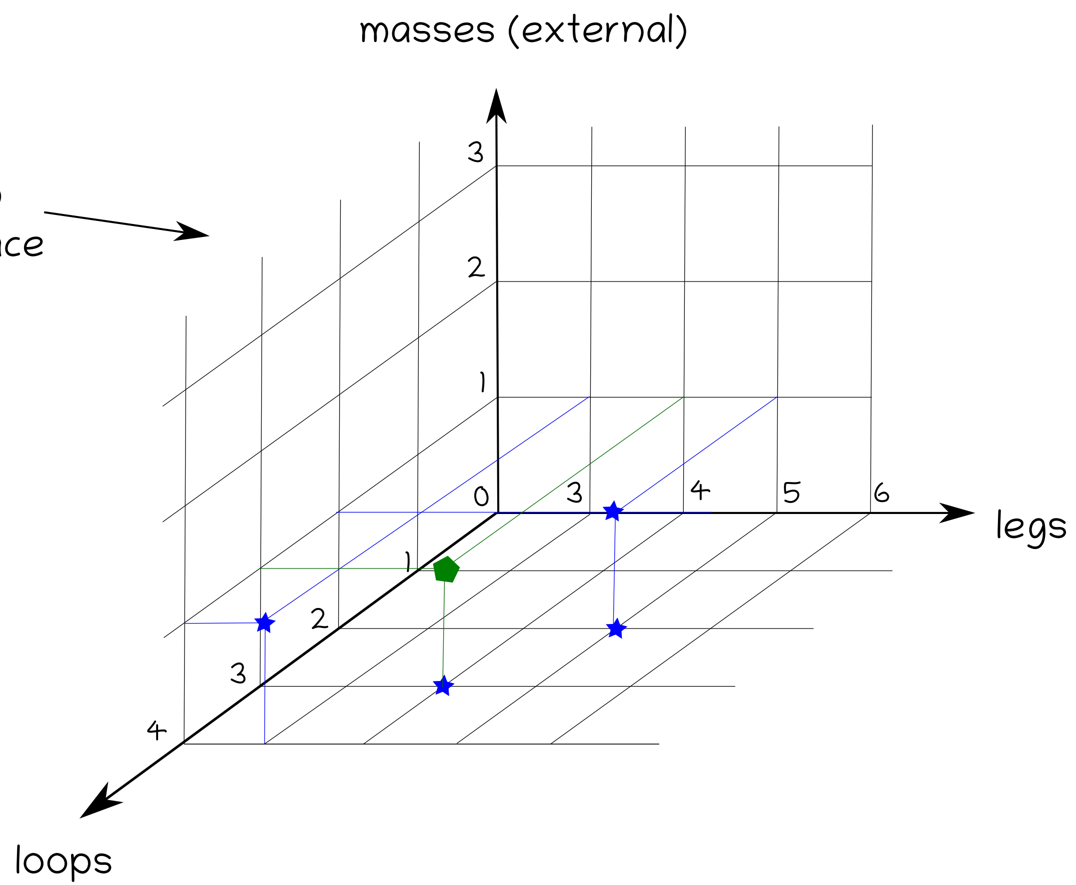

★ Done

⬟ In-sight (X+jet @3-loop)

Mostly manageable with analytical methods

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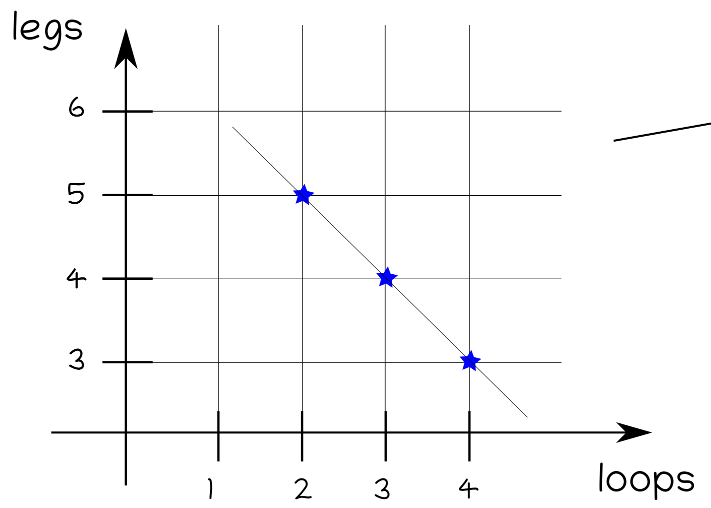

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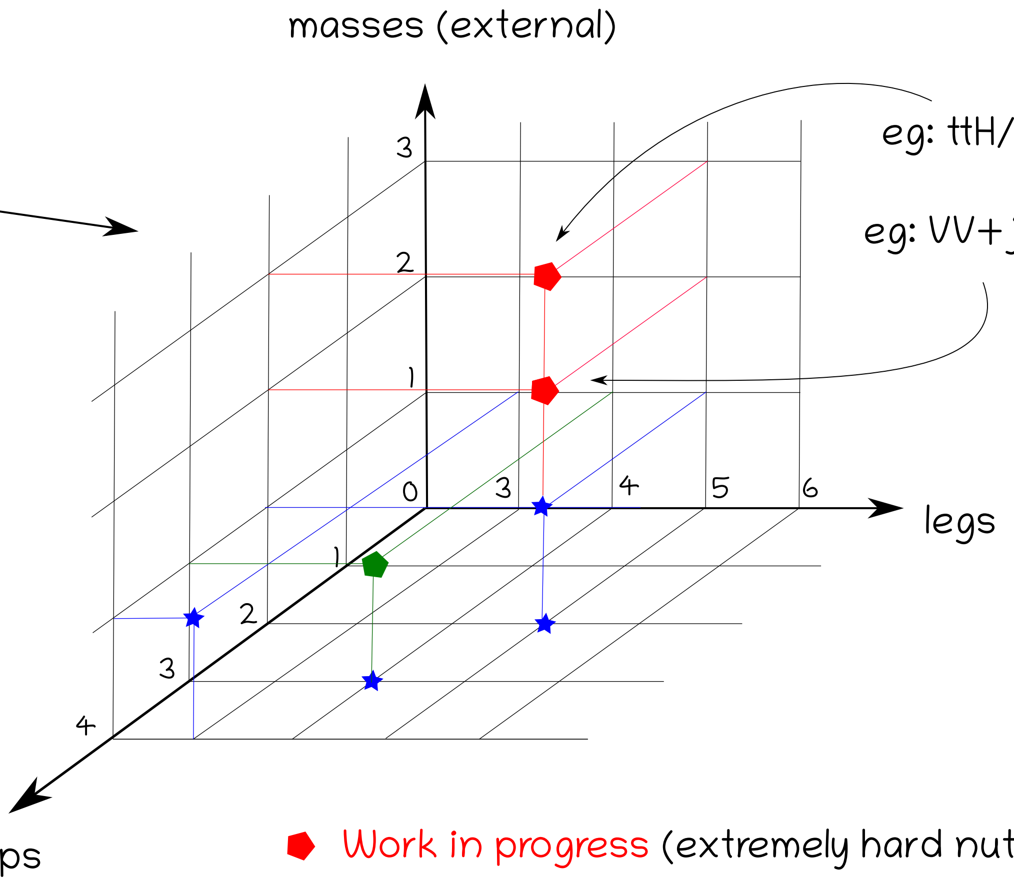

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⬟ In-sight (X+jet @3-loop)

Mostly manageable with analytical methods

⬟ Work in progress (extremely hard nuts to crack)

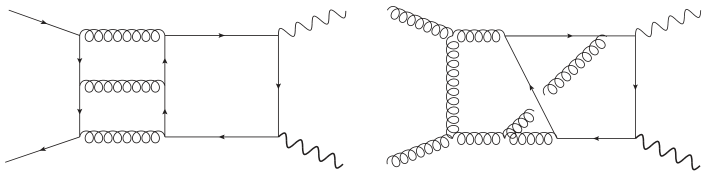
Numerical methods will play a dominant role

Scattering amplitudes: 2→2 @ 3-loops in QCD

All 3-loop 2→2 amplitudes with external massless partons are now available

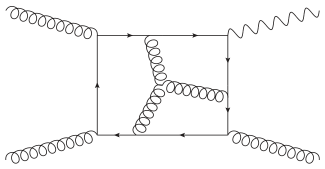
Master Integrals [Henn, Mistlberger, Wasser '20] + Calculation of the amplitudes [Bargiela, Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi '21,'22]

diphoton



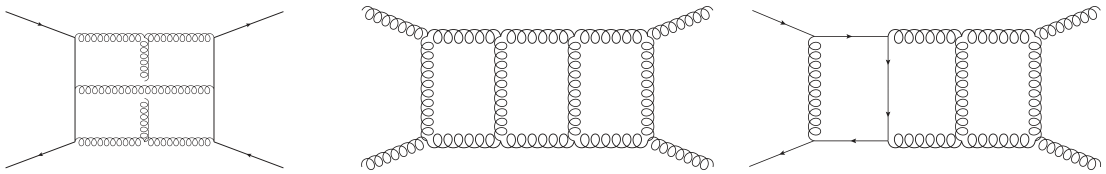
N^{3}LO pheno application in sight

single photon



next-to-hardest

dijet



devising a subtraction scheme
huge challenge + open questions

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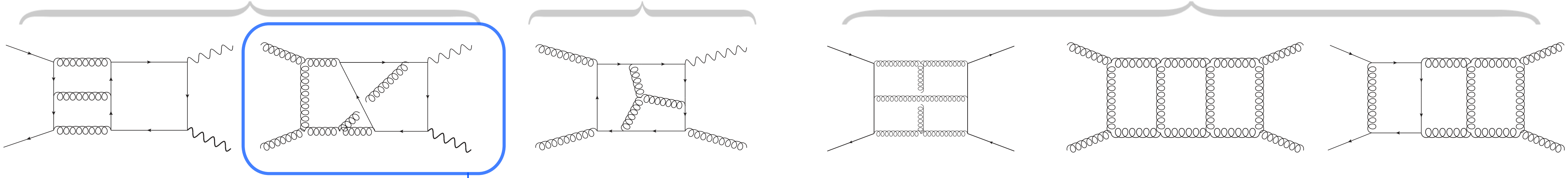
diphoton | single photon | dijet



N^3LO pheno application in sight

next-to-hardest

devising a subtraction scheme
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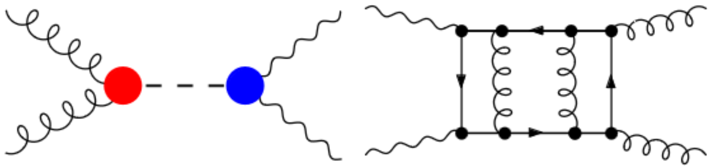
First pheno application of a 3-loop QCD amplitude

Signal-background interference in Higgs-mediated diphoton production [Bargiela, FB, Caola, Devoto, von Manteuffel, Tancredi '23]



Interesting effects:

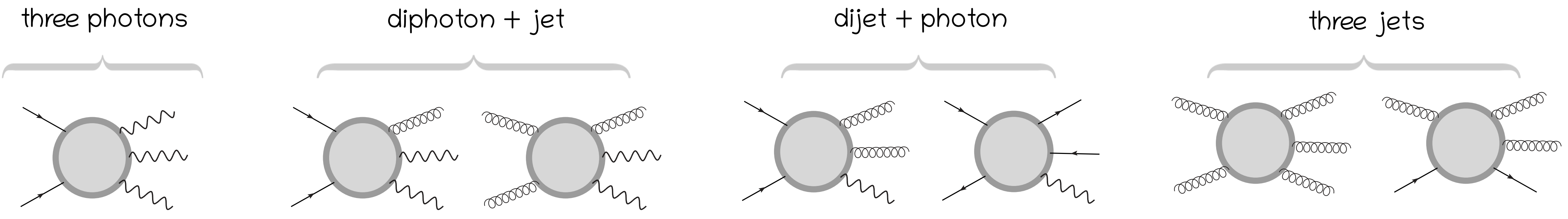
- Apparent mass shift O(50-80 MeV)
- Destructive interference effects ~ 1.6% reduction of signal XS
- Looser indirect bounds on Γ_H

Scattering amplitudes: 2→3 @ 2-loops in QCD

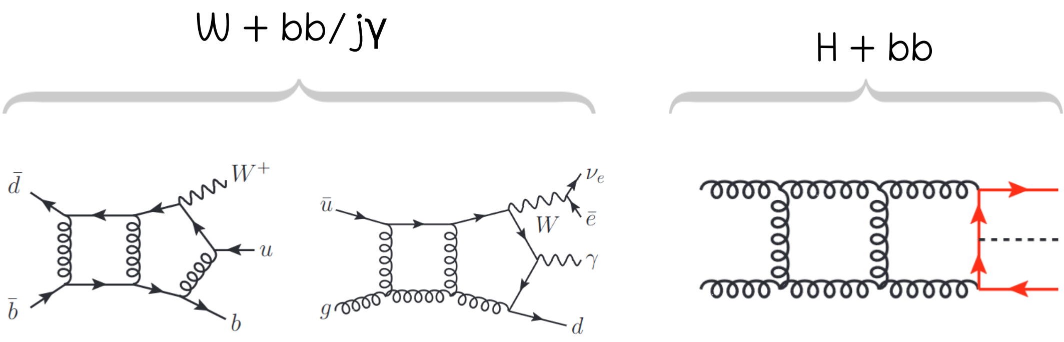
Contributors: [Abreu, Agarwal, Badger, FB, Chawhdry, Chicherin, Czakon, Cordero Febres, Gehrmann, Brønnum-Hansen, Hartanto, Henn, Ita, Klinkert, Kryś, Marcoli, Mitov, Moodie, Page, Pascual, Peraro, Poncelet, Sotnikov, Tancredi, Manteuffel von, Zoia]

All 2→3 massless amplitudes basically available (some in leading-colour approx. some exact)

Big boost from availability & fast evaluation of "Pentagon Functions" [Chicherin, Sotnikov '20] + new methods to cope with algebraic complexity



First results and steady progress on 2→3 amplitudes with one massive external particle (all available in leading-colour)



One-mass Pentagon functions [Chicherin, Sotnikov, Zoia '22]

Looking ahead

- two external masses, eg tt+j/γ, VV+j/γ
- three external masses, eg tt+H/W

Promising numerical methods:

- pySecDec [Borowka, Heinrich, Jahn, Jones, Kerner, Langer, Magerya, Poldaru, Schlenk, Villa, Zirke]
- DiffExp [Hidding] + Seasyde [Armadillo, Bonciani, Devoto, Rana, Vicini]
- AMFlow [Chen,Liu, Ma, Tao, Zhang]

questions: efficiency, grids (?)

Subtraction frameworks

IR-subtraction schemes

Two main approaches:

Local subtraction

$$\int |\mathcal{M}|^2 F_J \,\mathrm{d}\Phi^{(d)} = \int \left(|\mathcal{M}|^2 F_J - S\,\mathrm{d}\Phi^{(4)}\right) + \int S\mathrm{d}\Phi^{(d)}$$

Antenna [Gehrmann-de Ridder, Gehrmann, Glover et al]
Stripper [Czakon, Heymes, Mitov, Poncelet]
Nested Soft Collinear [Caola, Melnikov, Röntsch]
Projection-2-Born [Cacciari, Dreyer, Karlberg, Salam, Zanderighi]
Local analytic [Magnea, Maina, Pelliccioli, Signorile-Signorile, Torrielli, Uccirati]
Colorfull [Del Duca, Duhr, Kardos, Somogyi, Trocsanyi]

Slicing

$$\int |\mathcal{M}|^2 F_J \,\mathrm{d}\Phi^{(d)} = \int_0^{\Lambda} |\mathcal{M}|^2 F_J \,\mathrm{d}\Phi^{(d)} + \int_{\Lambda}^{1} S\mathrm{d}\Phi^{(4)} + \mathcal{O}(\Lambda)$$

q_T [Catani, Grazzini], [Bozzi, Catani, Grazzini, Ferrera, de Florian, Cieri, Devoto, Mazzitelli, Sargsyan, Torre]
N-jettiness [Boughezal, Liu, Petriello, Ellis, Campbell, Williams; Tackmann, Gaunt, Stahlhofen, Walsh]
k_t-ness [Buonocore, Grazzini, Haag, Rottoli, Savoini]

Both approches (in various incarnations) have proved very successfull

Key requirements to success:

- Subtraction: deal with singularities of multi-particle amplitudes: Sector decomposition vs Exploit property of amplitudes
- Slicing: good slicing parameter Λ + ability to express the cross section for $\Lambda \to 0$

- Devising a NNLO IR subtraction scheme is not a (conceptual) bottleneck anymore
- In practice we are far from an NLO-like situation: completely generic, simple and automatable framework (e.g. CS, FKS)

IR subtraction schemes: Slicing

Example: N-jettines

$$\lim_{\tau\to 0} d\sigma^{N^k LO}_{pp\to V+X} \approx B \otimes B \otimes S \otimes H \otimes d\sigma^{LO}_{pp\to V}$$

[Stewart, Tackmann, Waalewijn]

q_T

Similar factorisation + resummation formula

[Collins, Soper, Sterman] [Catani, De Florian, Grazzini] [Becher, Neubert]

NNLO:

- N-jettiness: amenable to describe low jet-multiplicity cross-sections: 0/1 jets + colour singlets
- q_T: developed for colour-singlet, extended to heavy-quarks pair and recently heavy-quarks + col. singlet

see M. Grazzini's talk on Friday

Pros: 1) implementation less involved → publicly available programs

Matrix [Grazzini, Kallweit, Wiesemann] MCFM [Campbell, Ellis, Neumann, Williams]

2) offer natural matching procedure (NNLO-PS)

MINNLO-PS (q_T) [Nason, Lombardi, Mazzitelli, Monni, Re, Wiesemann, Zanderighi]

GENEVA (N-jettiness) [Alioli, Bauer, Berggren, Guns,Tackmann, Walsh]

N^3LO:

Slicing methods conceptually simpler than subtraction: easier to extend to N^3LO

V+j @NNLO & knowledge (factorisation/resummation) of cross-section at small values of the slicing parameter

N-jettiness

- N^3LO Beam-functions [Ebert, Mistlberger, Vita '20] [Baranowski, Behring, Melnikov, Tancredi, Wever '22]
- 0-jettiness soft function, first results [Baranowski, Delto, Melnikov, Wang '22]
- The higher the jettiness, the (much) harder the soft function

q_T

All ingredients for colour singlet @N^3LO available

- N^3LO Soft function [Li, Zhu '16]
- N^3LO Beam-functions [Ebert, Mistlberger, Vita '20]

IR subtraction scheme: local subtractions

Fully local, therefore very efficient (although comparison between methods not really well defined)

- a given approach to the various ensuing aspects/complexities define a local subtraction scheme

Conceptually (rather) insensitive to jets multiplicity, but implementation highly non-trivial

- Frontier of jets multiplicity: 3 jets @ NNLO [Czakon, Mitov, Poncelet '22]
- In principle, we can now address any final state signature/multiplicity

X+jet @ NNLO
key ingredient for X@N^{3}LO

Most robust and well developed subtraction schemes:

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Antenna subtraction

- complete pheno for X+jet @NNLO
- full-colour dijet@NNLO [Chen, Huss, Gehrmann, Glover, Mo '22]
- automation for 3-jets, gluonic case [Chen, Huss, Gehrmann, Glover, Marcoli '22]
- first ingredients for N^{3}LO antennae from inclusive colour-singlet decays at N^{3}LO [Chen, Jakubčik, Marcoli, Stagnitto '23]

Sector-Improved Residue Subtraction (Stripper)

- full-colour dijet@NNLO [Czakon, van Hameren, Mitov, Poncelet, '19]
- vast 2→3 pheno studies@NNLO [Czakon, Mitov, Poncelet]
- three-jets @NNLO [Czakon, Mitov, Poncele '21]

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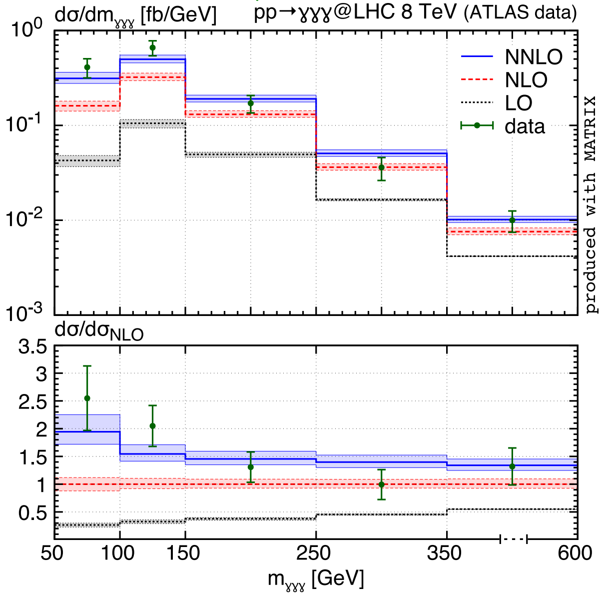
Sector-Improved Residue Subtraction (Stripper)

> A central goal of the present work is to demonstrate the feasibility of three-jet hadron collider computations with NNLO precision.
>
> On the technical side, the enormous computational cost of the present calculation ($\sim 10^6$ CPUh) makes it clear that further refinements in the handling of real radiation contributions to NNLO calculations are desirable.
>
> [Czakon, Mitov, Poncelet '22]

Selection of recent phenomenological studies

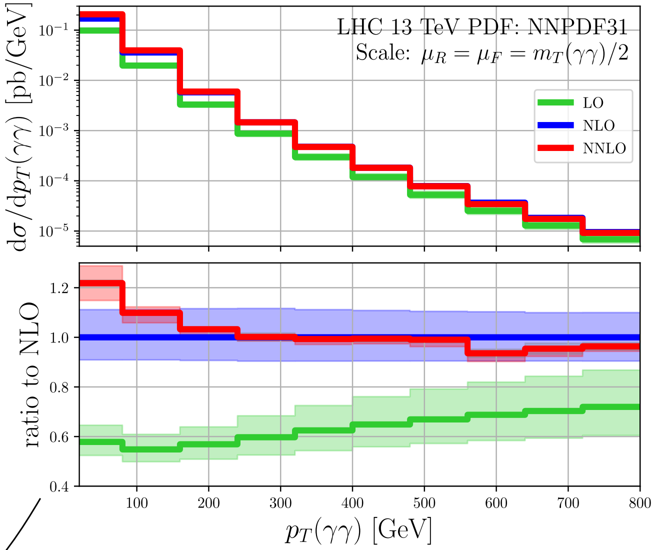
Fixed-order predictions: 2→3 massless (aka jets) @ NNLO

three-photon



[Kallweit, Sotnikov, Wiesemann '20]

diphoton + jet

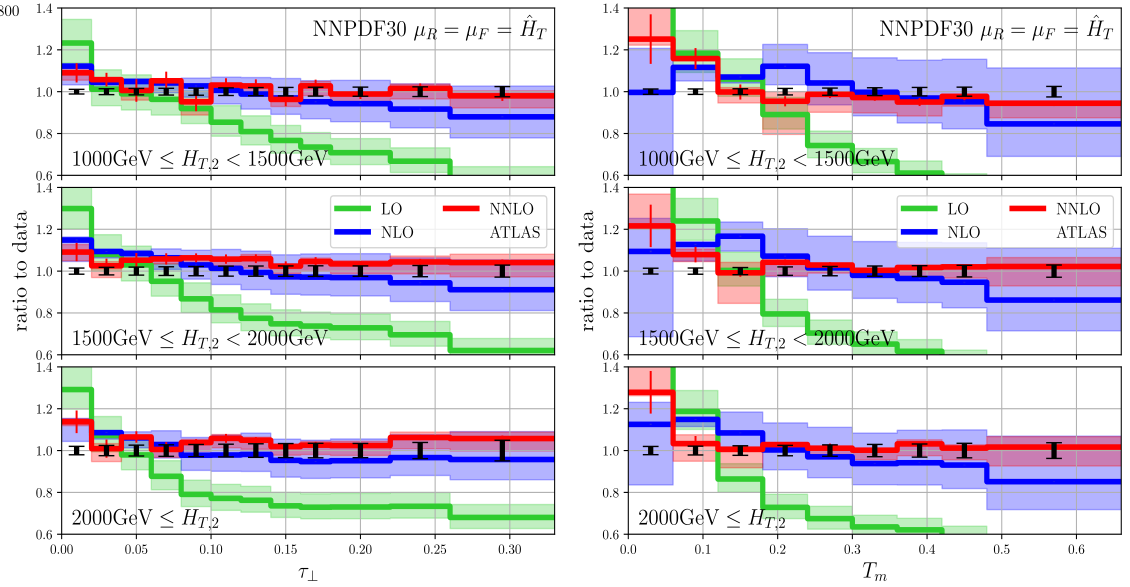


[Chawdhry, Czakon, Mitov, Poncelet '21]

Impressive progress on 2→3 cross sections:

- three-photons [Chawdhry, Czakon, Mitov, Poncelet '19] [Kallweit, Sotnikov, Wiesemann '20]
- diphoton+jet: qq/qg [Chawdhry, Czakon, Mitov, Poncelet '21]
- diphoton+jet: gg [Badger, Gehrmann, Marcoli, Moodie '21]
- dijet+photon: [Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet, Zoia '23]
- three-jets: [Czakon, Mitov, Poncelet '21]

three-jet production



[Czakon, Mitov, Poncelet '22] [Alvarez, Cantero, Czakon, Llorente, Mitov, Poncelet '23]

Outlook: diphoton+jet + qT → diphoton@N^{3}LO

Rich and interesting pheno: 3-jets

Jet rates

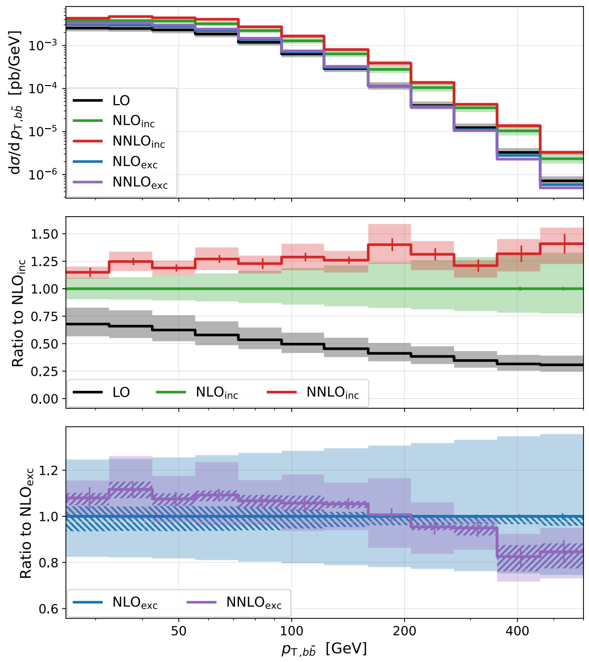
$$R_{3/2}(X,\mu_R,\mu_F) = \frac{d\sigma_3(\mu_R,\mu_F)/dX}{d\sigma_2(\mu_R,\mu_F)/dX}$$

Event shapes @ LHC

$$T_\perp = \frac{\sum_i |\vec{p}_{T,i}\cdot\hat{n}_\perp|}{\sum_i |\vec{p}_{T,i}|} \quad T_m = \frac{\sum_i |\vec{p}_{T,i}\times\hat{n}_\perp|}{\sum_i |\vec{p}_{T,i}|}$$

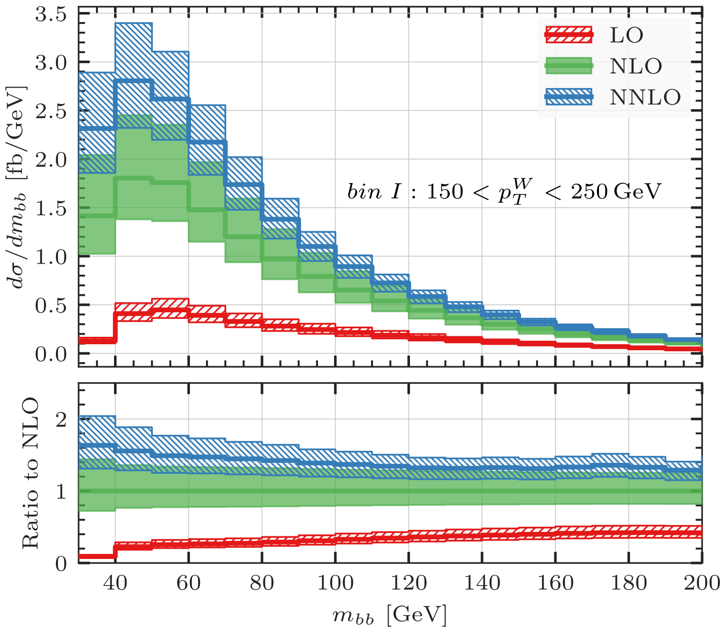
Fixed-order predictions: 2→3 associated productions @ NNLO

W+bb (m_b=0)



[Hartanto, Poncelet, Popescu, Zoia, '22]

W+bb (m_b≠0)



[Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini '22]

2L amplitudes:

- W+bb (m_b≠0): Wbb (m_b=0) + "massification"
  ε^{-1}_{coll}→ log(m_b/Q)
- ttH: 2-loop tt + soft Higgs, p_H → 0

→ qT soft-function for 2 massive radiators recoiling against X

ttH @ NNLO

σ [pb]	$\sqrt{s} = 13$ TeV	$\sqrt{s} = 100$ TeV
σ_{LO}	$0.3910^{+31.3\%}_{-22.2\%}$	$25.38^{+21.1\%}_{-16.0\%}$
σ_{NLO}	$0.4875^{+5.6\%}_{-9.1\%}$	$36.43^{+9.4\%}_{-8.7\%}$
σ_{NNLO}	$0.5070\,(31)^{+0.9\%}_{-3.0\%}$	$37.20(25)^{+0.1\%}_{-2.2\%}$

[Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Savoini '23]

- mb=0: no large log(m_b/p_T), but flavour tagging non-trivial
- mb≠0: flavoured jet tagging unambiguous, potentially large log(m_b/p_T)

see G. Stagnitto's talk on flavour tagging

see J. Mazzitelli's talk (Fri)

4FS vs 5FS:
generally good agreement, 4FS lower cross-section
improved by change of scheme: PDFS + α_S

N^3LO for colour singlet: fully differential predictions

Higgs production at N^3LO [Chen, Gehrmann, Glover, Huss, Mistlberger, Pelloni '21]

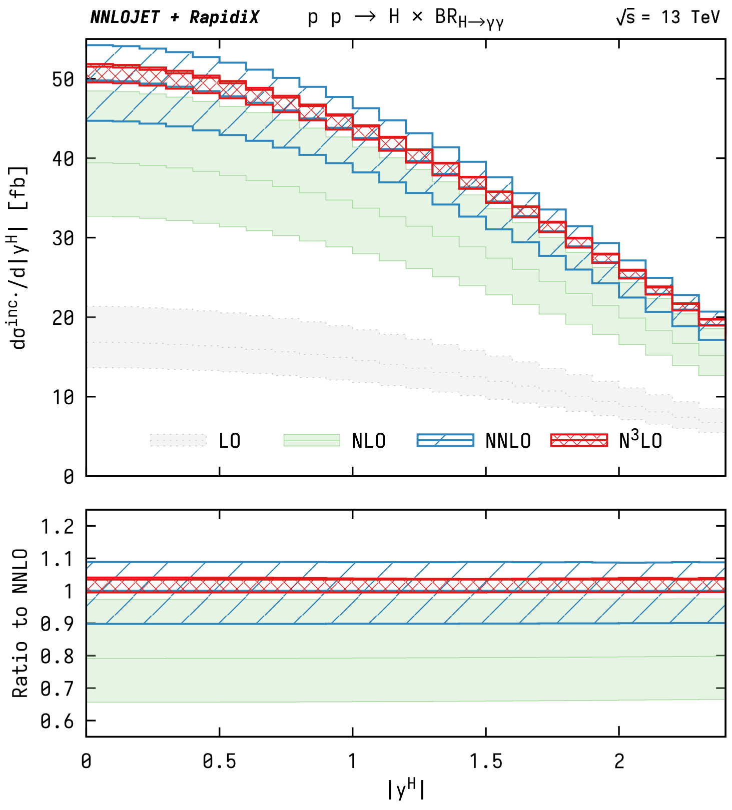
Run-time is that of a Higgs+jet@NNLO

Higgs+jet@NNLO + Projection 2 Born

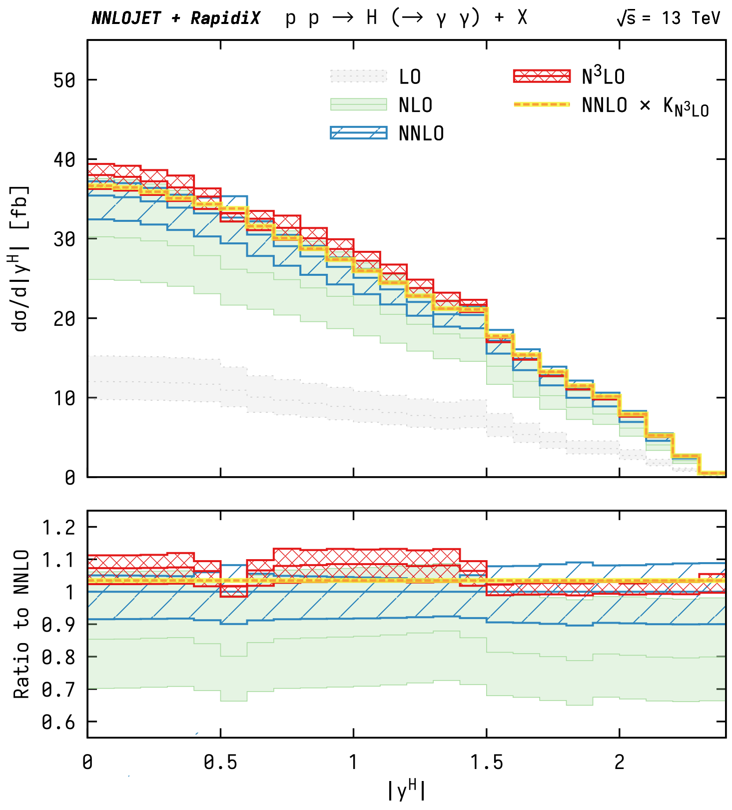
relies on calculation of incluse Higgs rapidity distribution [Dulat, Mistlberger, Pelloni '18]

Highly efficient

Inclusive



Fully Differential



$p_T^{\gamma_1} > 0.35 \cdot m_{\gamma\gamma}$

$p_T^{\gamma_2} > 0.25 \cdot m_{\gamma\gamma}$

$|y^\gamma| < 2.37$

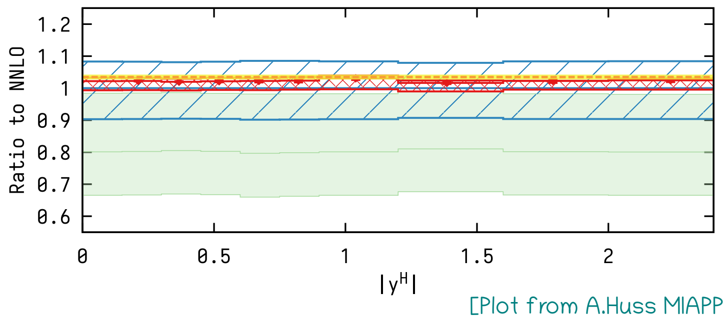
reject $1.37 < |y^\gamma| < 1.52$

photon isolation in $\Delta R < 0.2$

$\hookrightarrow \sum_{\Delta R_{i\gamma}<0.2} p_{T,i} < 0.05$

Product cuts on photons pT [Salam, Slade '21]

$\sqrt{p_T^{\gamma_1} p_T^{\gamma_2}} \geq 0.35 \cdot M_H$



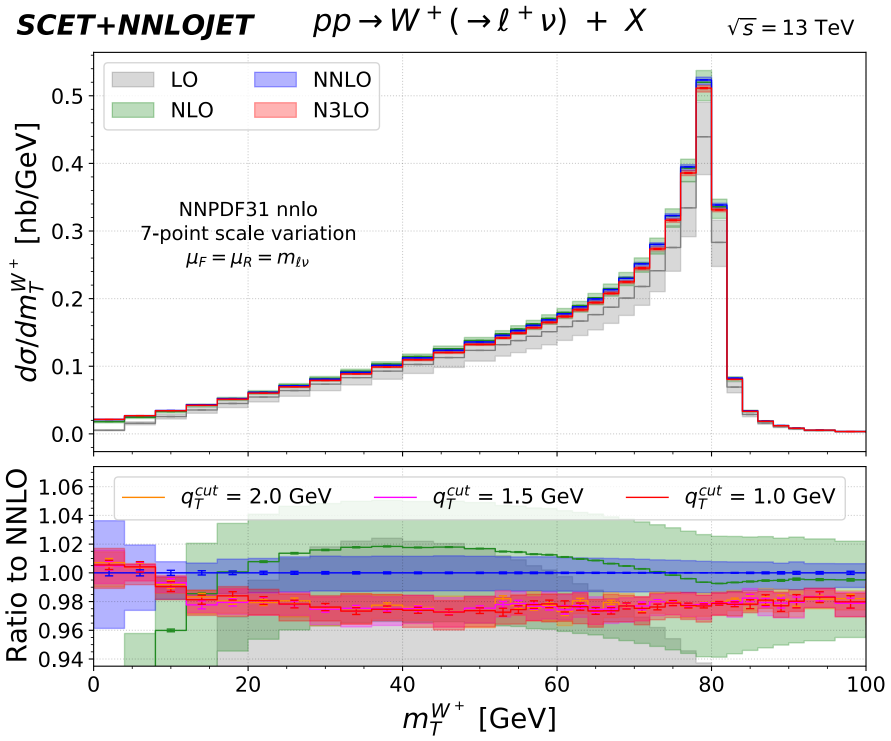
[Plot from A.Huss MIAPP '22]

N^3LO for colour singlet: fully differential predictions

N^3LO predictions for vector boson production: W/Z

V+jet @ NNLO + q_T to achieve N^3LO to guarantee good stability + insensitivity to slicing cut: few million CPU hours

W@N^3LO

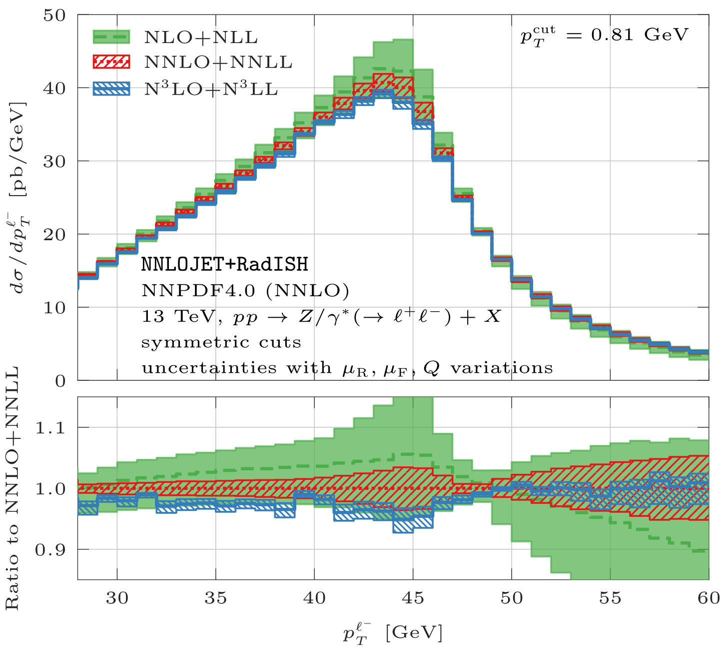


[Chen, Gehrmann, Glover, Huss, Yang, Zhu '22]

Transverse mass + charge asymmetry

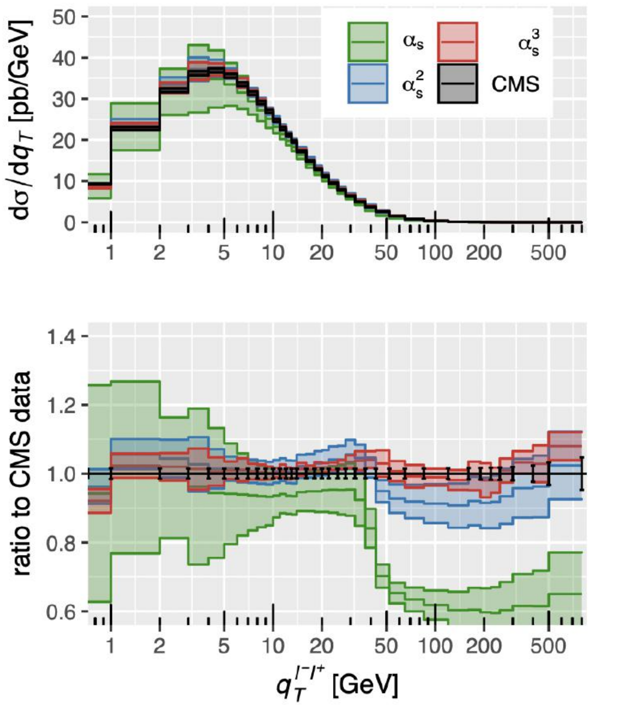
Z@N^3LO + Resummation

$N^3LO + N^3LL$



[Chen, Gehrmann, Glover, Huss, Monni, Rottoli, Re, Torrielli '22]

$N^3LO + N^4LL$ [Neumann, Campbell, '22]



Resummation is crucial for reliable description of transverse observables

more in T. Neumann's talk (Fri)

PDFs and four-loop splitting functions

For consistent N^{3}LO predictions: N^{3}LO PDFs are needed

First approximate N^{3}LO PDFs set [J. McGowan, T. Cridge, L. A. Harland-Lang, R. S. Thorne '22]

see Cruz-Martinez's talk for status on PDFs

see G. Zanderighi's talk for pheno discussion

$$Q^2 \frac{\mathrm{d} f_i(x,Q^2)}{\mathrm{d}Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} P_{ij}(\alpha_s, z) f_j\left(\frac{x}{z}, Q^2\right)$$

$$P_{ij}(\alpha_s, z) = P_{ij}^{(0)}(z) + a P_{ij}^{(1)}(z) + a^2 P_{ij}^{(2)}(z) + \underbrace{a^3 P_{ij}^{(3)}(z)}_{\text{N}^3\text{LO splitting kernels}} \qquad a = \frac{\alpha_s}{2\pi}$$

$i = g,\, q,\, \bar{q}$

N^{3}LO splitting kernels → Effectively a 4-loop calculation

Problem conveniently formulated in Mellin space:

$$\gamma_{ij}^{(k)}(N) = -\int_0^1 x^{N-1} P_{ij}^{(k)}(z)$$

—— work very hard ⟶ Inverse Mellin transformation to get result in z-space

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Methods to get the anomalous-dimension:

- Forward Compton scattering $\gamma(q) + p(k) \to \gamma(q) + p(k)$
- Renormalization of twist-2 operators/ Operator Matrix ELements (OMEs)

  Promising approach in covariant gauge [Gehrmann, von Manteufell, Yang '23]

Current status and results:

- (Non-)singlet P_{qq} large n_f [Davies, Ruijl, Vogt, Ueda, Vermaseren '16] [Basdew-Sharma, Pelloni, Herzog, Vogt '22]
- Non-singlet P_{qq} large N_c + approx. subleading [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]
- Pure singlet P_{qq} [Falcioni, Herzog, Moch, Vogt '23]

→ basically complete the calculation of P_{qq}

Summary and outlook

Fast and steady progress on fixed-order predictions

Large class of processes are now accessible with higher than NLO accuracy

New studies for processes with diverse and high-multiplicities at higher order published monthly

Take-home regarding methods:

✓ Great progress on multiloop amplitudes for massless processes, enablig broad pheno studies

✓ Subtraction methods can now address any signature. Slicing is pushing the N^3LO frontier

✗ Multiloop amplitudes with many scales are hard. Current bottleneck, need efficient, modern numerical methods

✗ Story not yet over for subtraction: high-jet multiplicities or N^3LO slicing very expensive. Lots of room for improvements

Outlook: selected suggestions on what to expect for the near future:

- Zbb@NNLO (same technology as Wbb)
- Colour singlets @N^3LO: ZH, WH and $\gamma\gamma$

and farther future:

- Jet cross sections @N^3LO

But clever ideas are around the corner!